A Note on Reaction-Diffusion Systems with Skew-Gradient Structure

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Abstract

Reaction-diffusion systems with skew-gradient structure can be viewed as a sort of activator-inhibitor systems. We use variational methods to study the existence of steady state solutions. Furthermore, there is a close relation between the stability of a steady state and its relative Morse index. Some numerical results will also be discussed.

1 Introduction

In this note we consider reaction-diffusion systems of the form

\begin{align}
M_1 u_t &= D_1 \Delta u + F_1(u,v), \\
M_2 v_t &= D_2 \Delta v - F_2(u,v),
\end{align}

(1.1)

\(x \in \Omega, \ t > 0.\) (1.2)

Here \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), \(u(x,t)\) is an \(m_1\)-dimensional vector function, \(v(x,t)\) is an \(m_2\)-dimensional vector function, \(M_1, M_2, D_1\) and \(D_2\) are positive definite matrices, and there exists a function \(F\) such that \(\nabla F = (F_1, F_2)\). Such systems can be viewed as a sort of activator-inhibitor systems.

A well-known example is

\begin{align}
u_t &= d_1 \Delta u + f(u) - v, \\
\tau v_t &= d_2 \Delta v + \sigma u - \gamma v,
\end{align}

(1.3)

(1.4)

where \(d_1, d_2, \sigma, \gamma, \tau \in (0, \infty)\) and \(f\) is a cubic polynomial. The case of \(d_2 = 0\) has been con-

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sidered as a model for the Hodgkin-Huxley system [13, 22] to describe the behavior of electrical impulses in the axon of the squid. More recently, several variations of this system appeared in neural net models for short-term memory and in nerve cells of heart muscle.

As in [29], (1.1)-(1.2) will be referred as a skew-gradient system in which a steady state is a critical point of

\[ \Phi(u, v) = \int_{\Omega} \frac{1}{2} (D_1 \nabla u, \nabla u) - \frac{1}{2} (D_2 \nabla v, \nabla v) - F(u, v) \, dx. \quad (1.5) \]

A steady state \((\bar{u}, \bar{v})\) is called a mini-maximizer of \(\Phi\) if \(\bar{u}\) is a local minimizer of \(\Phi(\cdot, \bar{v})\) and \(\bar{v}\) is a local maximizer of \(\Phi(\bar{u}, \cdot)\). It has been shown [29] that non-degenerate mini-maximizers of \(\Phi\) are linearly stable. This result gives a natural generalization of a stability criterion for the gradient system in which all the non-degenerate local minimizers are stable steady states.

A remarkable property proved in [29] is that any mini-maximizer must be spatially homogeneous if \(\Omega\) is a convex set. This kind of results have been established by Casten and Holland [5] and Matano [20] for the scalar reaction-diffusion equation, and generalized by Jimbo and Morita [15] and Lopes [19] for the gradient system. In case \(\Omega\) is symmetric with respect to \(x_j\), Lopes [19] showed that a global minimizer of gradient system is symmetric with respect to \(x_j\); while Chen [7] obtained parallel results for the global mini-maximizers in the skew-gradient system.

In connection with calculus of variations, there is a close relation between the stability of a steady state of skew-gradient system and its relative Morse index. Based on this idea, some stability criteria for the steady states of (1.1)-(1.2) are illustrated in section 2. In section 3, variational arguments are used to study the existence of steady states and their relative Morse indices. Section 4 contains numerical investigation of skew-gradient systems. A particular example to be studied is

\[
\begin{align*}
    u_t &= d_1 u_{xx} + f(u) - v - w, \\
    \tau_2 v_t &= d_2 v_{xx} + u - \gamma_2 v, \\
    \tau_3 w_t &= d_3 w_{xx} + u - \gamma_3 w,
\end{align*}
\]

which served as a model [4] for gas-discharge systems.

2 Stability Criteria

Let \(E\) be a Hilbert space. For a closed subspace \(U\) of \(E\), \(P_U\) denotes the orthogonal projection
from \( E \) to \( U \) and \( U^\perp \) denotes the orthogonal complement of \( U \). For two closed subspaces \( U \) and \( W \) of \( E \), denoted by \( U \sim W \) if \( P_U - P_W \) is a compact operator. In this case, both \( W \cap U^\perp \) and \( W^\perp \cap U \) are of finite dimensional. The relative dimension of \( W \) with respect to \( U \) is defined by

\[
\dim(W, U) = \dim(W \cap U^\perp) - \dim(W^\perp \cap U). \quad (2.1)
\]

If \( A \) is a self-adjoint Fredholm operator on \( E \), there is a unique \( A \)-invariant orthogonal splitting

\[
E = E_+(A) \oplus E_-(A) \oplus E_0(A)
\]

with \( E_+(A), E_-(A) \) and \( E_0(A) \) being respectively the subspaces on which \( A \) is positive definite, negative definite and null. For a pair of self-adjoint Fredholm operators \( A \) and \( \tilde{A} \), it will be denoted by \( A \sim \tilde{A} \) if \( E_-(A) \sim E_-(\tilde{A}) \). In this case, a relative Morse index \( i(A, \tilde{A}) \) is defined by

\[
i(A, \tilde{A}) = \dim(E_-(\tilde{A}), E_-(A)). \quad (2.2)
\]

We refer to [1] for more details of relative Morse index.

For a critical point \((\tilde{u}, \tilde{v})\) of \( \Phi \), let \( \Phi''(\tilde{u}, \tilde{v}) \) denote the second Frechet derivative of \( \Phi \) at \((\tilde{u}, \tilde{v})\). A critical point \((\tilde{u}, \tilde{v})\) is called non-degenerate if the null space of \( \Phi''(\tilde{u}, \tilde{v}) \) is trivial. Let

\[
M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad D = \begin{pmatrix} -D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad Q = \begin{pmatrix} -I_{m_1} & 0 \\ 0 & I_{m_2} \end{pmatrix}
\]

and \( I_k \) be the \( k \times k \) identity matrix. For a gradient system, it is known that a non-degenerate critical point with non-zero Morse index is an unstable steady state. The next theorem gives a parallel result for the skew-gradient system.

**Theorem 1.** Suppose \( i(-Q, \Phi''(\tilde{u}, \tilde{v})) \neq 0 \) and \( \dim E_0(\Phi''(\tilde{u}, \tilde{v})) = 0 \), then for any positive definite matrices \( M_1 \) and \( M_2 \), \((\tilde{u}, \tilde{v})\) is an unstable steady state of (1.1)-(1.2).

In [29] Yanagida pointed out an interesting property that a non-degenerate minimaximizer of \( \Phi \) is always stable for any positive matrices \( M_1 \) and \( M_2 \) given in (1.1)-(1.2). An interesting question is whether there exist steady states with stability depending on the reaction rates of the system. Let \( P^+ \) and \( P^- \) be the orthogonal projections from \( E \) to \( E_+(Q) \) and \( E_-(Q) \) respectively. Define \( \Psi_0 = M^{-\frac{1}{2}}(D\Delta - \nabla^2 F(\tilde{u}, \tilde{v}))M^{-\frac{1}{2}}, \psi_1 = P^-\Psi_0P^- \)
and $\psi_2 = P^+\Psi_0P^+$. Set $m = m_1 + m_2$, $\mathcal{D} = H^2(\Omega, \mathbb{R}^m)$, $\rho_i(\psi_1) = \inf_{z \in \mathcal{D}} \frac{\langle \psi_1 z, z \rangle}{\|P^-z\|_{L^2}^2}$ \hspace{1cm} (2.3)

and $\rho_s(\psi_2) = \sup_{z \in \mathcal{D}} \frac{\langle \psi_2 z, z \rangle}{\|P^+z\|_{L^2}^2}$. \hspace{1cm} (2.4)

**Theorem 2.** Assume that $i(-Q, \Phi''(\bar{u}, \bar{v})) = 0$ and $\dim E_0(\Phi''(\bar{u}, \bar{v})) = 0$. Then $(\bar{u}, \bar{v})$ is stable if $\rho_i(\psi_1) > \rho_s(\psi_2)$.

**Remark.** In case we treat the Dirichlet boundary condition $u|_{\partial \Omega} = v|_{\partial \Omega} = 0$, $\mathcal{D}$ is replaced by $H^2(\Omega, \mathbb{R}^m) \cap H_0^1(\Omega, \mathbb{R}^m)$.

The proofs of Theorem 1 and Theorem 2 can be found in [8].

### 3 Applications of Theorem 1 and Theorem 2

In dealing with a strongly indefinite functional $\Phi$, a critical point theorem established by Benci and Rabinowitz [3] can be used to obtain steady states of (1.1)-(1.2).

**Theorem 3.** Let $E$ be a separable Hilbert space with an orthogonal splitting $E = W_+ \oplus W_-$, and $B_r = \{\xi|\xi \in E, \|\xi\| < r\}$. Assume that $\Phi(\xi) = \frac{1}{2}\langle \hat{\Lambda}\xi, \xi \rangle + b(\xi)$, where $\hat{\Lambda}$ is a self-adjoint invertible operator on $E$, $b \in C^2(E, \mathbb{R})$ and $b'$ is compact. Set $S = \partial B_r \cap W_+$ and $N = \{\xi^- + se|\xi^- \in B_r \cap W_-, s \in [0, \bar{R}]\}$, where $e \in \partial B_1 \cap W_+$, $r > 0$ and $\bar{R} > \rho > 0$. If $\Phi$ satisfies $(PS)^*$ condition and $\sup_{\partial N} \Phi < \inf_S \Phi$, then $\Phi$ possesses a critical point $\bar{\xi}$ such that $\inf_S \Phi \geq \Phi(\bar{\xi}) \geq \sup_{\partial N} \Phi$. Moreover, if $W_\perp \sim E_\perp$, then

$$i(\hat{\Lambda}, \Phi''(\bar{\xi})) \leq \dim(W_-, E_-) + 1 \leq i(\hat{\Lambda}, \Phi''(\bar{\xi})) + \dim E_0(\Phi''(\bar{\xi})).$$ \hspace{1cm} (3.1)

**Remark.** (a) See e.g. [2, 8] for the definition of $(PS)^*$ condition.

(b) The index estimates (3.1) were obtained by Abbondandolo and Molina [2].

In a demonstration of using Theorem 3 to study the existence and stability of steady state solutions, we consider a perturbed FitzHugh-Nagumo system in the first example:

$$u_t = d_1\Delta u + f(u) - v, \hspace{1cm} (3.2)$$

$$\tau v_t = d_2\Delta v + u - \gamma v - h(v). \hspace{1cm} (3.3)$$

A steady state of (3.2)-(3.3) is a critical point

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Let $h$ satisfies the following condition:

$\{ \gamma > 9(2\beta^2 - 5\beta + 2)^{-1}, \text{and } h \text{ satisfies the following condition:} \}

(h1) $h \in C^1$, $h(0) = h'(0) = 0$ and $yh(y) \geq 0$ for $y \in \mathbb{R}$.

Define

$$
\Lambda = \begin{pmatrix}
-d_1 \Delta - f'(0) & 1 \\
1 & d_2 \Delta - \gamma
\end{pmatrix}.
$$

Let

$$
\mu_+^k = \frac{1}{2} \left[ (d_1 - d_2) \lambda_k - (f'(0) + \gamma) \\
+ \sqrt{(d_1 + d_2) \lambda_k - f'(0) + \gamma)^2 + 4} \right]
$$

and

$$
\mu_-^k = \frac{1}{2} \left[ (d_1 - d_2) \lambda_k - (f'(0) + \gamma) \\
- \sqrt{(d_1 + d_2) \lambda_k - f'(0) + \gamma)^2 + 4} \right],
$$

where $\{-\lambda_k\}$ are the eigenvalues of the Laplace operator and $\{ \phi_k \}$ are the corresponding eigenfunctions. By straightforward calculation

$$
\Lambda e_+^k \phi_k = \mu_+^k e_+^k \phi_k \text{ and } \Lambda e_-^k \phi_k = \mu_-^k e_-^k \phi_k,
$$

where

$$
e_+^k = (1, \frac{1}{2} [\sqrt{(d_1 + d_2) \lambda_k - f'(0) + \gamma)^2 + 4} \\
- [(d_1 + d_2) \lambda_k - f'(0) + \gamma])],
$$

$$
e_-^k = (1, -\frac{1}{2} [(d_1 + d_2) \lambda_k - f'(0) + \gamma \\
+ \sqrt{(d_1 + d_2) \lambda_k - f'(0) + \gamma)^2 + 4}].
$$

It is clear that $\mu_+^k > 0$ and $\mu_-^k < 0$ for all $k \in \mathbb{N}$.

Let $E_+ = \bigoplus_{k=1}^{\infty} V_+^k$ and $E_- = \bigoplus_{k=1}^{\infty} V_-^k$, where

$V_+^k = \{ s \phi_k e_+^k | s \in \mathbb{R} \}$ and $V_-^k = \{ s \phi_k e_-^k | s \in \mathbb{R} \}$. Define $\Lambda^+ = \Lambda|_{E_+}$, $\Lambda^- = \Lambda|_{E_-}$ and

$$
\langle \Lambda z_1, z_2 \rangle = \int_{\Omega} \left( (\Lambda^+)^{\frac{1}{2}} z_1, (\Lambda^+)^{\frac{1}{2}} z_2 \right) \\
- \left( (\Lambda^-)^{\frac{1}{2}} z_1, (\Lambda^-)^{\frac{1}{2}} z_2 \right) dx
$$

for $z_1, z_2 \in E$. As an application of Theorem 3, we have the following result.

**Theorem 4.** Let $B_R$ be a ball in $\mathbb{R}^n$ with radius $R$. If $\Omega$ contains a ball $B_R$ with $R$ being sufficiently large, then there exists a steady state $(\bar{u}, \bar{v})$ of (3.2)-(3.3), and

$$
\Phi''(\bar{u}, \bar{v}) \leq 1 \leq i(-Q, \Phi''(\bar{u}, \bar{v})) + \dim E_0(\Phi''(\bar{u}, \bar{v})).
$$

In view of Theorem 1, $(\bar{u}, \bar{v})$ is unstable if it is a non-degenerate critical point of $\Phi$. More details can be found in [8].
We now turn to some examples to seek stable steady states of skew-gradient systems. Consider
\[ u_t = \Delta u - u - v, \quad (3.5) \]
\[ \tau v_t = \Delta v + 2v + u - |v|v. \quad (3.6) \]

Straightforward calculation gives
\[ \Lambda = \begin{pmatrix} -\Delta + 1 & 1 \\ 1 & \Delta + 2 \end{pmatrix}, \]
\[ \mu_j^+ = \frac{1}{2}(3 + \sqrt{(2\lambda_k - 1)^2 + 4}) \quad \text{and} \quad \mu_k^- = \frac{1}{2}(3 - \sqrt{(2\lambda_k - 1)^2 + 4}). \]

It is clear that \( \mu_k^- > 0 \) for all \( k \in \mathbb{N} \). Suppose \( \Omega \) is a bounded domain in which the eigenvalue distribution of the Laplace operator (under homogeneous Dirichlet boundary conditions) satisfies the following property:
\[ \lambda_1 < \frac{1}{2}(\sqrt{5} + 1) < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \cdots \]

Then it is easily seen that \( \mu_1^- > 0 \), and \( \mu_k^- < 0 \) if \( k \geq 2 \). It follows that \( i(-Q, \Lambda) = -1 \).

**Theorem 5.** There is a non-constant steady state \((\bar{u}, \bar{v})\) of (3.5)-(3.6). Moreover, if \( \dim(\Phi''(\bar{u}, \bar{v})) = 0 \) and \( \tau \geq \frac{2-\lambda_1}{1+\lambda_1} \), then \((\bar{u}, \bar{v})\) is stable.

In the next example, consider (1.3)-(1.4) with \( f(u) = \alpha u - u^3 \) and \( \sigma = 1 \). Suppose there is a \( j \in \mathbb{N} \) such that if
\[ d_1\lambda_j + \frac{1}{d_2\lambda_j + \gamma} < \alpha < \inf\{d_1\lambda_k + \frac{1}{d_2\lambda_k + \gamma} | k \in \mathbb{N}\{j\}\} \quad (3.7) \]

By direct calculation \( \mu_j^+ < 0 \) and \( \mu_k^- > 0 \) for \( k \in \mathbb{N}\{j\} \). Also, \( \mu_k^- < 0 \) for all \( k \in \mathbb{N} \). Hence \( i(-Q, \Lambda) = 1 \). Applying Theorem 3 yields a steady state \((\bar{u}, \bar{v})\) of (1.3)-(1.4). Furthermore,
\[ i(-Q, -\Phi''(\bar{u}, \bar{v})) \leq 0 \leq i(-Q, -\Phi''(\bar{u}, \bar{v})) + \dim E_0(\Phi''(\bar{u}, \bar{v})). \]

This implies that \( i(-Q, -\Phi''(\bar{u}, \bar{v})) = 0 \) if \((\bar{u}, \bar{v})\) is a non-degenerate critical point of \( \Phi \). Then by Theorem 2, \((\bar{u}, \bar{v})\) is stable if \( \tau < \frac{2}{\alpha} \). In case of dealing with homogeneous Neumann boundary conditions, \((\bar{u}, \bar{v})\) is a spatially inhomogeneous steady state if (3.7) holds for \( j \geq 2 \). In other words, there exists a stable pattern for (1.3)-(1.4).

For the FitzHugh-Nagumo system, the steady state solutions satisfy
\[ d_1\Delta u + f(u) - v = 0, \quad (3.8) \]
\[ \frac{d_2}{\sigma}\Delta v + u - \gamma \frac{1}{\sigma}v = 0, \quad (3.9) \]

where \( f(u) = (1-u)(u-\beta)u, \beta \in (0, \frac{1}{2}) \). If \( \mathcal{L} = \sigma^{-1}(-d_2\Delta + \gamma)^{-1} \) under homogeneous Dirichlet (respectively Neumann) boundary conditions,
then for any critical point \( \bar{u} \) of
\[
\psi(u) = \int_{\Omega} \left( \frac{d_1}{2} |\nabla u|^2 + uL \right) - \int_0^u f(\zeta) d\zeta \, dx,
\]
\( (\bar{u}, L\bar{u}) \) is a steady state of FitzHugh-Nagumo system. In view of the fact that \( \sigma \int_{\Omega} uL u \, dx = \int_{\Omega} d_2 |\nabla v|^2 + \gamma v^2 \, dx \), it is easily seen that \( \psi \) is bounded from below. In addition to minimizers, the Mountain Pass Lemma has been used to obtain non-trivial solutions \([9, 10, 11, 17, 21, 24, 28, 32] \) of (3.8)-(3.9)

Let \( u \) be a critical point of \( \psi \). Straightforward calculation yields
\[
\psi''(u) = -\Delta + L - f'(u),
\]
where \( \psi'' \) is the second Frechet derivative of \( \psi \) and the Morse index of \( u \) will be denoted by \( i_s(\psi''(u)) \). On the other hand, \( (u, L u) \) is also a critical point of
\[
\Phi(u, v) = \int_{\Omega} \left( \frac{d_1}{2} |\nabla u|^2 - \frac{d_2}{2\sigma} |\nabla u|^2 + uv \right.
\]
\[
- \frac{\gamma}{2\sigma} v^2 - \int_0^u f(\xi) d\xi \, dx.
\]

**Proposition 1.** If \( u \) is a critical point of \( \psi \) and \( v = Lu \), then
\[
\dim E_0(\psi''(u)) = \dim E_0(\Phi''(u, v))
\]
and
\[
i_s(\psi''(u)) = i(-Q, \Phi''(u, v)).
\]

We refer to [8] for a proof of Proposition 1.

For a critical point \( u \) obtained by the Mountain Pass Lemma, it is known [6] that
\[
i_s(\psi''(u)) \leq 1 \leq i_s(\psi''(u)) + \dim E_0(\psi''(u)).
\]
Then by Proposition 1
\[
i(-Q, \Phi''(u, Lu)) \leq 1 \leq i(-Q, \Phi''(u, Lu)) + \dim E_0(\Phi''(u, Lu)).
\]
Thus if \( \dim E_0(\psi''(u)) = 0 \), it follows from Theorem 1 that \( (u, Lu) \) is an unstable steady state of (1.3)-(1.4).

Let \( \hat{\psi}_1 = P^- (D\Delta - \nabla^2 F(\bar{u}, \bar{v}))P^- \), \( \hat{\psi}_2 = P^+ (D\Delta - \nabla^2 F(\bar{u}, \bar{v}))P^+ \),
\[
\rho_i(\hat{\psi}_1) = \inf_{z \in D} \frac{\langle \hat{\psi}_1 z, z \rangle_{L^2}}{\| P^- z \|_{L^2}^2}
\]
and
\[
\rho_s(\hat{\psi}_2) = \sup_{z \in D} \frac{\langle \hat{\psi}_2 z, z \rangle_{L^2}}{\| P^+ z \|_{L^2}^2}
\]

**Theorem 6.** Assume that \( i(-Q, \Phi''(\bar{u}, \bar{v})) = 0 \) and \( \dim E_0(\Phi''(\bar{u}, \bar{v})) = 0 \). Then \( (\bar{u}, \bar{v}) \) is stable if one of the following conditions holds:

(i) \( \rho_i(\hat{\psi}_1) > 0, \rho_s(\hat{\psi}_2) \geq 0 \) and
\[
\frac{\rho_s(\hat{\psi}_2)}{\rho_i(\hat{\psi}_1)} < \| M_2^{-1} \|^{-1} \| M_1 \|^{-1}.
\]

(ii) \( \rho_i(\hat{\psi}_1) \leq 0, \rho_s(\hat{\psi}_2) < 0 \) and
\[
\frac{\rho_i(\hat{\psi}_1)}{\rho_s(\hat{\psi}_2)} < \| M_1^{-1} \|^{-1} \| M_2 \|^{-1}.
\]
Theorem 6 directly follows from Theorem 2. We refer to [8] for the detail.

If $u$ is a non-degenerate minimizer of $\psi$ and $v = L u$, then Proposition 1 implies that $i(-Q, \Phi(u, v)) = 0$. Notice that

$$D\Delta - \nabla^2 F(u, v) =\begin{pmatrix} -d_1\Delta - f'(u) & 1 \\ 1 & d_2\Delta - \frac{\gamma}{\sigma} \end{pmatrix},$$

Since $f'(\xi) = -3\xi^2 + 2(\beta + 1)\xi - \beta \leq (\beta^2 - \beta + 1)/3$, it easy to check that $\rho_i(\hat{\psi}_1) = \rho_i(-d_1\Delta - f'(u)) \geq d_1\lambda_1 - \frac{(\beta^2 - \beta + 1)}{3}$ and $\rho_s(\hat{\psi}_2) = \rho_s\left(\frac{d_2\Delta - \frac{\gamma}{\sigma}}{\sigma}\right) \leq -(d_2\lambda_1 + \gamma)/\sigma$, where $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k < \cdots$ are the eigenvalues of $-\Delta$. If $\rho_i(\hat{\psi}_1) \leq 0$ and $\tau < \frac{3(d_2\lambda_1 + \gamma)}{\sigma((\beta^2 - \beta + 1) - d_1\lambda_1)}$, condition (ii) of Theorem 6 holds and consequently $(u, v)$ is a stable steady state of (1.3)-(1.4).

4 Numerical Results

We report some numerical work on the skew-gradient systems, and compare with the theoretical results.

We start with the following reaction-diffusion system:

$$\begin{align*}
    u_t &= d_1u_{xx} + u(u - \beta)(1 - u) - v - w, \\
    \tau_2v_t &= d_2v_{xx} + u - \gamma_2v, \\
    \tau_3w_t &= d_3w_{xx} + u - \gamma_3w,
\end{align*}$$

where $\beta = 0.3$, $\gamma_2 = 1$, $\gamma_3 = 20$, and the homogeneous Neumann boundary conditions will be under consideration. In (4.1)-(4.3), $u$ can be viewed as an activator while $v$ and $w$ act as inhibitors. In view of the theoretical results mentioned in the previous sections, we look for the pattern formation for (4.1)-(4.3) in case the diffusion rate of the activator is small ($d_1 = 10^{-6}$).

By taking $d_2 = 1$ and $d_3 = 10^{-6}$, various types of spatially inhomogeneous steady states have been observed through numerical calculation. In Figure 1 and Figure 3, there is one peak on the profile of $u$; the one in Figure 1 is symmetric with respect to the spatial variable, while the other is not. We found also instances of steady states with two peaks on the profile.
of $u$; but the distance between peaks can be different. We remark based on numerical observation that, with $\tau_2 = \tau_3 = 10^{-4}$, such inhomogeneous steady states are stable under the flow generated by (4.1)-(4.3). Moreover, the solution profiles tell that $w$ is roughly equal to $\gamma_3^{-1} u$ in magnitude.

![Figure 1: solution profile of $u$](image1)

Figure 1: solution profile of $u$

![Figure 2: profiles of $v$ and $w$](image2)

Figure 2: profiles of $v$ and $w$

![Figure 3: solution profile of $u$](image3)

Figure 3: solution profile of $u$

![Figure 4: profiles of $v$ and $w$](image4)

Figure 4: profiles of $v$ and $w$

![Figure 5: solution profile of $u$](image5)

Figure 5: solution profile of $u$
We next turn to the case when both inhibitors $v$ and $w$ are acting with large diffusion ($d_2 = d_3 = 1$). As show in Figure 9-10, the pulse (or peak of $u$) becomes wider. The fact that $\gamma_3 > \gamma_2$ results in $v > w$.

Keeping $d_3 = 1$ and reducing $d_2$ to $10^{-1}$, we obtain a stable steady state with rather different profiles as shown in Figure 11-12.
4.2

In this subsection we come back to the reaction-diffusion system

\begin{align*}
  u_t &= u_{xx} - u - v, \\
  \tau v_t &= v_{xx} + 2v + u - |v|v, \\
  x &\in (0, 3), t > 0, \\
  u(0, t) &= v(0, t) = u(3, t) = v(3, t) = 0.
\end{align*}

As we know from Theorem 5, the choice of \( \tau = 0.1 \) leads the flow converging to a non-constant steady state (Figure 13). The behavior in the phase plane of the state variables, at the midpoint of the domain \( (x = 1.5) \), exhibits a spiral-inward convergence (Figure 14).

On the other hand, we conjecture that such a non-constant steady state become unstable if the value of \( \tau \) is taking much smaller. Indeed, when \( \tau = 0.005 \), we observed a time-periodic attractor (Figure 15-16).
The convergence history of the two calculated state variables is recorded in Figure 17-18, which strongly suggests the existence of a stable time-periodic solution. The change of stability seems to result from a Hopf bifurcation and deserves further investigation.

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